§5. Differential calculus on $\mathbb{R}$

§5.1 Differential and differentiation rules

Let $\Omega \subset \mathbb{R}$ be open (i.e. of type (a,b)),
$f: \Omega \to \mathbb{R}$, $x_0 \in \Omega$.

Definition 5.1:

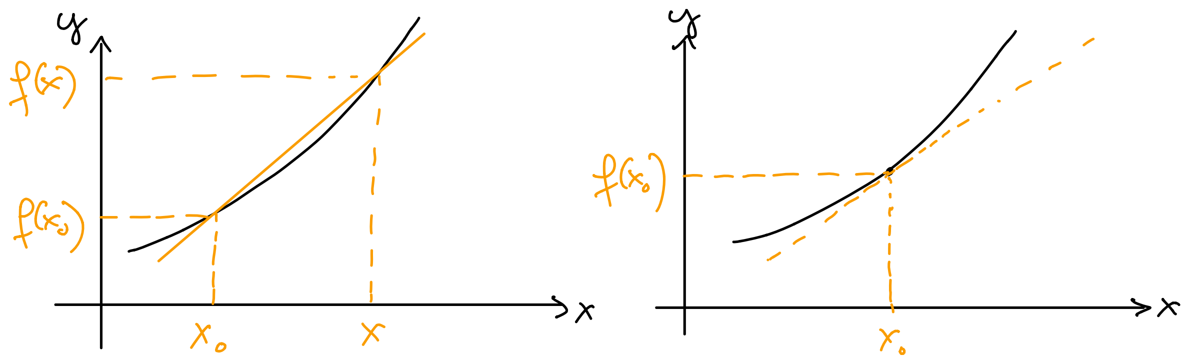
f is called "differentiable" at x_0, if the limit

$$\lim_{\substack{x \to x_0, \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0) =: \frac{df}{dx}(x_0)$$

exists. In this case we denote by $f'(x_0)$ the "derivative" or the "differential" of f at x_0.

Remark 5.1:

Geometrically, the "differential quotient" $\frac{f(x) - f(x_0)}{x - x_0}$ corresponds to the slope of the secant through the points $(x, f(x))$, $(x_0, f(x_0))$ of the graph $\mathcal{G}(f)$, and the differential $f'(x_0)$ is is the slope of the tangent at $\mathcal{G}(f)$ in $(x_0, f(x_0))$.



Definition 5.2:

$f: \Omega \longrightarrow \mathbb{R}$ is called "differentiable on Ω", if f is differentiable at every $x_0 \in \Omega$.

Example 5.1:

i) Let $f(x) = mx + b$, $x \in \mathbb{R}$, with constants $m, b \in \mathbb{R}$.
Then we have :

$$\forall\ x \neq x_0 : \quad \frac{f(x) - f(x_0)}{x - x_0} = m;$$

$\Rightarrow$ f is differentiable at every $x_0 \in \mathbb{R}$ with $f'(x_0) = m$.

ii) The function $f(x) = |x|$, $x \in \mathbb{R}$, is not differentiable at $x_0 = 0$, as

$$\lim_{x \uparrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \uparrow 0} \frac{|x|}{x} = -1 \qquad \lim_{x\uparrow 0} := \lim_{\substack{x \to 0 \\ x < 0}}$$

$$\neq \lim_{x \downarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \downarrow 0} \frac{x}{x} = 1. \qquad \lim_{x \downarrow 0} := \lim_{\substack{x \to 0 \\ x > 0}}$$

iii) Let $f(x) = \mathrm{Exp}(x)$, $x \in \mathbb{R}$. With Example 4.9 ii) we have for $x_0 \neq x = x_0 + h \in \mathbb{R}$

$$\frac{\mathrm{Exp}(x_0+h) - \mathrm{Exp}(x_0)}{h} = \frac{\mathrm{Exp}(x_0)\left(\mathrm{Exp}(h)-1\right)}{h}$$

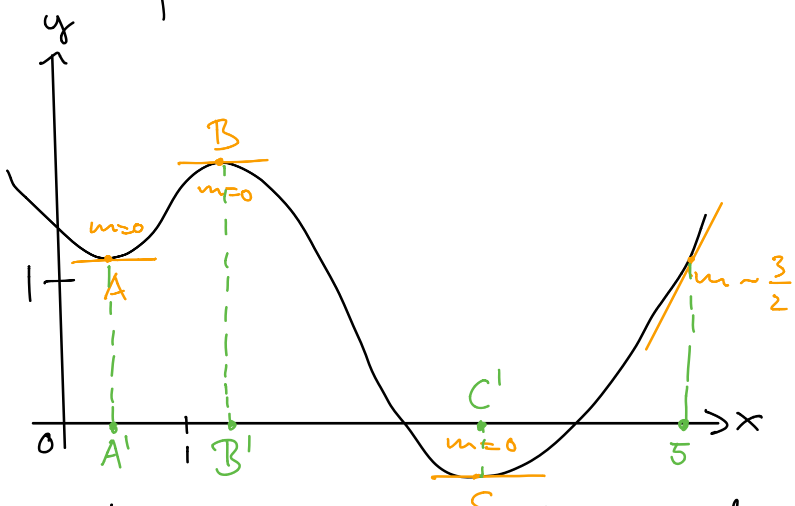
$$= \mathrm{Exp}(x_0) \sum_{k=1}^{\infty} \frac{h^{k-1}}{k!} \longrightarrow \mathrm{Exp}(x_0) \quad (h \to 0)$$

$\Rightarrow$ the function $\mathrm{Exp}: \mathbb{R} \to \mathbb{R}$ is differentiable at every $x_0 \in \mathbb{R}$ with $\mathrm{Exp}'(x_0) = \mathrm{Exp}(x_0)$,

or

$$\mathrm{Exp}' = \mathrm{Exp}$$

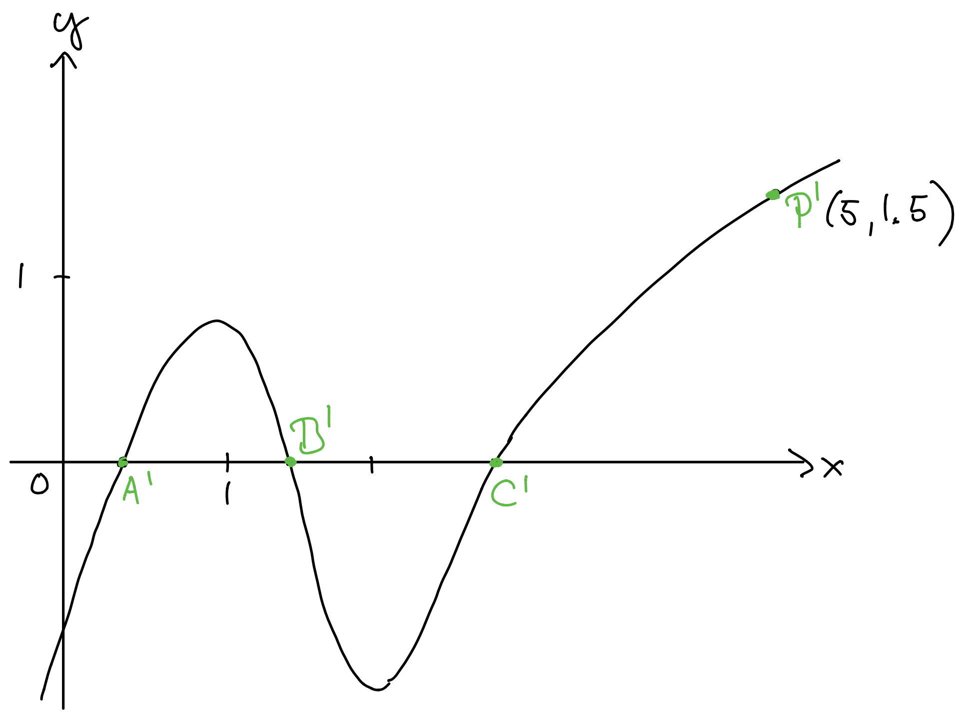
iv) The graph of the function f is given as follows:



What is the graph of the derivative?

Solution:

Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis:



Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

v) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

vi) If $f(x) = \sqrt{x}$, find the derivative of f.

solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is smaller than the domain of f which is $[0, \infty)$.

Proposition 5.1:

Is $f : \Omega \to \mathbb{R}$ differentiable at $x_0 \in \Omega$, then f is continuous at x_0.

Proof:

For $(x_k)_{k \in \mathbb{N}} \subset \Omega$ with $x_k \to x_0$ $(k \to \infty)$ we have according to Prop. 3.3:

$$f(x_k) - f(x_0) = \underbrace{\frac{f(x_k) - f(x_0)}{x_k - x_0}}_{\to f'(x_0) \text{ finite}} \cdot \underbrace{(x_k - x_0)}_{\to 0} \longrightarrow 0 \quad \begin{pmatrix} k \to \infty, \\ x_k \neq x_0 \end{pmatrix}$$

and $f(x_k) - f(x_0) = 0$, if $x_k = x_0$

Thus $f(x_k) \to f(x_0)$ $(k \to \infty)$, as desired $\square$

Remark 5.2:

i) Prop. 5.1 shows that differentiable functions are continuous. However, continuous functions are not necessarily differentiable as the example $f(x) = |x|$, $x \in \mathbb{R}$ shows (Example 5.1 ii))

ii) There exist continuous functions $f: \mathbb{R} \to \mathbb{R}$, which are nowhere differentiable (see exercises).

Proposition 5.2 (Differentiation laws):

Let $f, g : \Omega \to \mathbb{R}$ be differentiable at $x_0 \in \Omega$. Then the functions $f+g$, $f \cdot g$ and, if $g(x_0) \neq 0$, also the function f/g, are differentiable at x_0, and we have

i) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$,

ii) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$,

iii) $$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

Proof:

i) For $x \in \Omega$, $x \neq x_0$, Prop. 3.3 gives

$$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}$$

$$= \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}$$

$$\xrightarrow{(x \to x_0)} f'(x_0) + g'(x_0)$$

$\Rightarrow$ $f+g$ is differentiable at x_0 with:

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

ii) Analogously, Prop. 5.1 gives

$$\frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \frac{(f(x)-f(x_0))g(x)+f(x_0)(g(x)-g(x_0))}{x-x_0}$$

$$= \frac{f(x)-f(x_0)}{x-x_0}\, g(x) + f(x_0)\,\frac{g(x)-g(x_0)}{x-x_0}$$

$$\xrightarrow{(x \to x_0)} f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$

and thus the desired result for $f \cdot g$.

iii) Due to ii), it is sufficient to prove the case $f = 1$. Then Prop. 5.1 and $g(x_0) \neq 0$, imply that $g(x) \neq 0$ for all x in a neighborhood of x_0, and $g(x) \to g(x_0)$ $(x \to x_0, x \in \Omega)$.

Prop. 3.3 then gives

$$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \frac{g(x_0) - g(x)}{x - x_0} \, \frac{1}{g(x)g(x_0)}$$

$$\xrightarrow{(x \to x_0,\, x \neq x_0)} -\frac{g'(x_0)}{g^2(x_0)}$$

$\square$

<u>Example 5.2:</u>

i) For $n \in \mathbb{N}$, the function $f(x) = x^n$, $x \in \mathbb{R}$, is differentiable with $f'(x) = nx^{n-1}$.

<u>Proof:</u> (by induction)

$n = 1$: see Example 5.1 i)

$n \to n+1$: Set $f(x) = x^n$, $g(x) = x$. According to induction assumption, f and g are differentiable:

$$f'(x) = nx^{n-1}, \quad g'(x) = 1.$$

Prop. 5.2 ii) then gives

$$\frac{dx^{n+1}}{dx}(x) = (fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$= (n+1)x^n.$$

$\square$

ii) Polynomials $p(x) = a_n x^n + \dots + a_1 x + a_0$ are differentiable on $\mathbb{R}$ with

$$p'(x) = n a_n x^{n-1} + \dots + a_1 .$$

iii) Rational functions $r(x) = \frac{p(x)}{q(x)}$ are differentiable on their domain of definition

$$\Omega = \{ x \in \mathbb{R} \mid q(x) \neq 0 \} ,$$

and

$$r' = \frac{p'q - pq'}{q^2}$$

is again a rational function on Ω.

<u>Example 5.3</u> (from physics):

We let an object fall from an altitude $y_0 > 0$. Then its height will be the following function of time:

$y(t) = v_0 t + y_0 - \frac{1}{2} g t^2$, where

v_0: initial velocity (m/s)

$v(t)$: vertical velocity as function of time (m/s)

y_0: initial altitude (m)

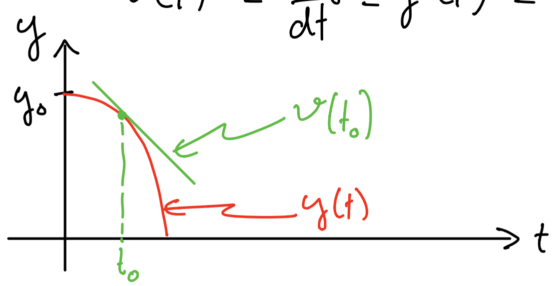
$y(t)$: altitude as function of time

t : time elapsed (s)

g : acceleration due to gravity ($9.81\ m/s^2$)

$\Longrightarrow$ velocity $v(t)$ is given by:

$$v(t) = \frac{dy}{dt} = y'(t) = v_0 - gt$$



Proposition 5.3 (Chain rule):

Let $f : \Omega \to \mathbb{R}$ be differentiable at $x_0 \in \Omega$, and let $g : \mathbb{R} \to \mathbb{R}$ be differentiable at $y_0 = f(x_0)$. Then the function $g \circ f : \Omega \to \mathbb{R}$ is differentiable at x_0, and we have

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$$

Example 5.4:

For all functions $f(x) = mx + c$, $g(y) = ly + d$ we have

$$(g \circ f)(x) = l(mx + c) + d = lmx + (lc + d),$$

and therefore $(g \circ f)'(x_0) = lm = g'(f(x_0)) \cdot f'(x_0)$.

Proof of Prop. 5.3:

For $x \in \Omega$ with $f(x) \neq f(x_0)$ write

$$\frac{(g\circ f)(x) - (g\circ f)(x_0)}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

Let $(x_k)_{k\in\mathbb{N}} \subset \Omega$ with $x_k \to x_0$ $(k \to \infty)$, and let $f(x_k) \neq f(x_0)$, $k \in \mathbb{N}$. According to Prop. 5.1 we have for $x_k \to x_0$ also $f(x_k) \to f(x_0)$ $(k \to \infty)$, and we get

$$\lim_{k\to\infty} \frac{(g\circ f)(x_k) - (g\circ f)(x_0)}{x_k - x_0} = g'(f(x_0))\, f'(x_0). \quad (*)$$

If for a sequence $x_k \to x_0$ $(k \to \infty)$ we have

$$x_k \neq x_0, \quad f(x_k) = f(x_0), \ k \in \mathbb{N},$$

then we get

$$f'(x_0) = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = 0,$$

and

$$\lim_{k\to\infty} \frac{g(f(x_k)) - g(f(x_0))}{x_k - x_0} = 0 = g'(f(x_0))\, f'(x_0).$$

Together with $(*)$ this then gives the desired convergence for every sequence $(x_k) \subset \Omega$ with $x_k \to x_0$ $(k \to \infty)$.

Example 5.5:

i) The function

$$x \longmapsto (x^3 + 4x + 1)^2 = x^6 + 8x^4 + 2x^3 + 16x^2 + 8x + 1$$

is of the form $g \circ f$ with

$$g(y) = y^2, \quad f(x) = x^3 + 4x + 1.$$

Example 5.2 i) and Prop. 5.3 then give

$$\frac{d}{dx}(x^3 + 4x + 1)^2 = \underbrace{2(x^3 + 4x + 1)}_{= g'(f(x))} \cdot \underbrace{(3x^2 + 4)}_{= f'(x)}$$

$$= 6x^5 + 32x^3 + 6x^2 + 32x + 8.$$

ii) The function $t \mapsto e^{\lambda t}$, where $\lambda \in \mathbb{R}$, is of the form $g \circ f$ with $g(x) = e^x$, $f(t) = \lambda t$

Together with Example 5.1 i) and iii) we get

$$\left.\frac{d}{dt}(e^{\lambda t})\right|_{t = t_0} = \underbrace{e^{\lambda t_0}}_{= g'(f(t_0))} \cdot \underbrace{\lambda}_{= f'(t_0)}.$$